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Energy-Momentum Four-Vector on a Wavefront

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We first discuss the synchronous and asynchronous interpretations of relativistic dynamics, and then prove that the synchronous formulation makes it possible to define in the limit $v \mapsto c$ the energy-momentum four-vector of a classical field on a wavefront propagating with the velocity of light. It is also shown that except for plane waves, the Poynting vector is not the energy flow vector of the electromagnetic field.

1. INTRODUCTION

Let us consider in classical field theory a boundary value problem with data on a wavefront propagating with the velocity of light (Hillion, 1988). We ask how to define the energy-momentum four-vector P_μ on this wavefront.

We start with a discussion on the definition of P_μ from the energy-momentum tensor $T_{\mu\nu}$ of the field. We use the signature $(+,+,+,+)$ with the coordinates x_i, $i = 1, 2, 3$, $x_4 = ict$ and the summation convention. The Greek indices take the values 1, 2, 3, 4; the Latin indices take the values 1, 2, 3.

2. TWO INTERPRETATIONS OF RELATIVISTIC DYNAMICS

2.1. The Question at Issue

As is well known, a particle of proper mass m with four-velocity u_μ has a momentum four-vector $P_\mu = (m/c^2)\gamma u_\mu$, where c is the velocity of light and $\gamma = (u_\mu u_\mu)^{-1/2}$. The extension of this formula to an arbitrary system is given in the rest frame (characterized by the superscript zero) of this

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system by the relation

$$P^0_\mu = \int g^0_\mu \, dV \tag{1}$$

where the integral is over the whole volume of the system at time $t = \text{const}$, while the energy-momentum density g^0_μ is defined in terms of $T^0_{\mu\nu}$ by the relations

$$g^0_j = \frac{i}{c} T^0_{j4}, \qquad g^0_4 = -T^0_{44} \tag{2}$$

This leads one to define P_μ in any inertial frame by the relation

$$P_\mu = \int g_\mu \, dV \tag{3}$$

The question at issue is: How does one define g_μ and dV so that P_μ is a four-vector? We start with a remark on the physical interpretation of the Lorentz transformations.

The principle of relativity imposes conditions which all physical laws have to satisfy. But these conditions take different forms according to the way that measurements are realized. There are two main points of view. The most usual one since Einstein's formulation of relativity is the synchronous interpretation in which one considers a system at a given moment in the observer's frame (Van Bladel, 1984). But some years ago, it became clear that the description is unfit for a global description of physical systems. In the asynchronous interpretation one looks at simultaneous events in the rest frame of the system (Van Bladel, 1984). These two interpretations do not supply the same relations for the transformations of the relativistic kinematics and dynamics under the Lorentz group. This leads to a different choice of variables associated to observable quantities. Consequently we shall discuss the definition (3) in both interpretations.

2.2. Synchronous Interpretation

In this interpretation each observer integrates at constant time in his inertial frame. Then the boundaries of the volume are taken as contemporary for both observer at rest and moving observer, so that the volume element transforms according to the relation

$$dV = \gamma^{-1} \, dV^0, \qquad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{4}$$

Then to make (4) consistent with (2) and (3), g_μ is defined by the relation

$$g^s_j = \frac{i}{c} T_{j4}, \qquad g^s_4 = -T_{44} \tag{5}$$

where the superscript s means that one uses the synchronous interpretation. This definition of the energy-momentum "vector" has three drawbacks.

(i) P_μ^s is a four-vector only for closed systems (Möller, 1957), that is, for systems satisfying the conservation equation $\partial_\nu T_{\mu\nu} = 0$. For nonclosed systems P_μ^s requires extra forces of dubious physical meaning, some of which are discussed by Gamba (1967) and Gron (1973).

(ii) The two quantities dV and dV^0 do not refer in fact to the same set of events.

(iii) If one has in the rest frame $T_{13}^0 \neq 0$ or $T_{23}^0 \neq 0$, then, as we shall see later, the energy-momentum vector for an observer moving in the z direction has a component normal to this direction different from zero, opposite to the prerelativistic meaning of the concept of momentum (Gron, 1973).

2.3. Asynchronous Interpretation

In fact, there is no reason why each observer should integrate at constant time. Then Gron (1973) gives a general method for dealing with bodies having relativistic speeds. We quote: "One starts with the theory for the description of the object in his inertial system. This is put in covariant form, i.e., it is given a tensor formulation. In this form the theory can be transformed to any inertial frame."

Consequently, one has to write (1) and (2) in a covariant form, and to do that we follow Rohrlich (1970). In Minkowski space the volume V^0 is the three-dimensional hyperplane $t = \text{const}$. It has a unit normal which is the timelike four-vector n^0 with components $n_\mu^0 = (0, 0, 0, i)$, so that $n_\mu n_\mu = -1$. In general, a spacelike hyperplane with normal vector n and $n_\mu n_\mu = -1$ has the infinitesimal element $d^3\sigma_\mu = n_\mu \, d^3\sigma$ and this measure has the following properties:

(i) $d^3\sigma_\mu$ is a four-vector.
(ii) $d^3\sigma = -n_\mu \, d^3\sigma_\mu$ is invariant, that is, $d^3\sigma = dV^0$.

Then the convariant form of (1), (2) valid for any referential frame is

$$P_\mu^a = \int g_\mu^a \, d^3\sigma, \qquad g_\mu^a = T_{\mu\nu} n_\nu \tag{6}$$

The superscript a means that one uses the asynchronous interpretation. We shall illustrate (6) in the next section.

Remark. Kwal (1949) seems to have been the first to promote the asynchronous interpretation of the Lorentz transformation in his relativistic discussion of the incorrect factor of 4/3 in the Abraham–Lorentz force equation.

3. THE ENERGY-MOMENTUM FOUR-VECTOR ON A WAVEFRONT

3.1. Lorentz Transformation of the Energy-Momentum Tensor

Let us consider an observer moving along $0z$ with the uniform velocity v and the corresponding Lorentz transformation:

$$L=\begin{vmatrix} I & 0 \\ 0 & M \end{vmatrix}, \qquad M=\gamma\begin{vmatrix} 1 & iv/c \\ -iv/c & 1 \end{vmatrix}, \qquad \gamma=\left(1-\frac{v^2}{c^2}\right)^{-1/2} \tag{7}$$

I is the 2×2 identity matrix. Let T be the matrix with elements $T_{\mu\nu}$; then we get, under the transformation (7),

$$T \mapsto T' = L^t T L \tag{8}$$

L^t is the transpose matrix. In particular, we get

$$\begin{aligned}
&T_{13}=\gamma\left(T'_{13}-\frac{iv}{c}T'_{14}\right) \qquad T_{23}=\gamma\left(T'_{23}-\frac{iv}{c}T'_{24}\right)\\
&T_{14}=\gamma\left(T'_{14}+\frac{iv}{c}T'_{13}\right) \qquad T_{24}=\gamma\left(T'_{24}+\frac{iv}{c}T'_{23}\right)\\
&T_{33}=\gamma^2\left[T'_{33}-\frac{iv}{c}(T'_{43}+T'_{34})-\frac{v^2}{c^2}T'_{44}\right]\\
&T_{34}=\gamma^2\left(\frac{iv}{c}T'_{33}+\frac{v^2}{c^2}T'_{43}+T'_{34}-\frac{iv}{c}T'_{44}\right)\\
&T_{43}=\gamma^2\left(\frac{iv}{c}T'_{33}+tT'_{43}+\frac{v^2}{c^2}T'_{34}-\frac{iv}{c}T'_{44}\right)\\
&T_{44}=\gamma^2\left[\frac{-v^2}{c^2}T'_{33}+\frac{iv}{c}(T'_{43}+T'_{34})+T'_{44}\right]
\end{aligned} \tag{9}$$

If the primed frame is the rest frame, these relations reduce to

$$\begin{aligned}
&T_{13}=\gamma T^0_{13} \qquad T_{23}=\gamma T^0_{23}\\
&T_{14}=\frac{iv}{c}\gamma T^0_{13} \qquad T_{24}=\frac{iv\gamma}{c}T^0_{23}\\
&T_{33}=\gamma^2\left(T^0_{33}-\frac{v^2}{c^2}T^0_{44}\right) \qquad T_{34}=\frac{iv}{c}\gamma^2(T^0_{33}-T^0_{44})\\
&T_{44}=i\gamma^2\frac{v}{c}(T^0_{33}-T^0_{44}) \qquad T_{44}=\gamma^2\left(-\frac{v^2}{c^2}T^0_{33}+T^0_{44}\right)
\end{aligned} \tag{10}$$

since one has $T^0_{k4} = T^0_{4k} = 0$. From (5) and (10) one sees at once that if $T^0_{13} \neq 0$, one has $g^s_1 \neq 0$ and similarly with T^0_{23} and g^s_2. As previously said, this result is a bitter pill.

Let us now consider the relation (6). The vector orthogonal to the hyperplace $z - ct = 0$ is $n_\mu = (0, 0, v, ic)$, so that we get

$$g^a_\mu = \frac{1}{c}\,(T_{\mu 3} v + icT_{\mu 4}) \tag{11}$$

and using (10), the components (11) become

$$g^a_1 = g^a_2 = 0, \qquad g^a_3 = \frac{v}{c}\, g^0_4, \qquad g^a_4 = g^0_4 \tag{12}$$

which provide a more gratifying result than g^s_μ.

Remark. The previous four-vector n_μ is not a unit vector, to make the limit $v \mapsto c$ consistent. Moreover, with this choice, $g^a_\mu = g^0_\mu$.

3.2. What Happens on a Wavefront?

We assume that the wavefront is the hyperplane $z\text{-}ct = 0$ propagating along $0z$ with velocity of light c. We first write the relations (9) in the two alternative forms

$$\begin{aligned} vT'_{13} + icT'_{14} &= ic\gamma^{-1} T_{14} \\ vT'_{23} + icT'_{24} &= ic\gamma^{-1} T_{24} \\ vT'_{33} + icT'_{34} &= icT_{34} + vT_{44} \\ vT'_{43} + icT'_{44} &= icT_{44} - vT_{34} \end{aligned} \tag{13a}$$

$$\begin{aligned} -vT_{13} + icT_{14} &= ic\gamma^{-1} T'_{14} \\ -vT_{23} + icT_{24} &= ic\gamma^{-1} T'_{24} \\ -vT_{33} + icT_{34} &= icT'_{34} - vT'_{44} \\ -vT_{43} + icT_{44} &= icT'_{44} + vT'_{34} \end{aligned} \tag{13b}$$

According to (11), the left-hand side of the relation (13a) [resp. (13b)] gives modulo c^{-1} the components of the energy-momentum density vector g^a_μ in the primed (resp. unprimed) frame with the relative velocity between the two frames given by the four-vector with components $(0, 0, \pm v, ic)$.

The interesting point is that the relations (13) hold for $v = c$ and become

$$\begin{aligned} T'_{13}+iT'_{14}&=0\\ T'_{23}+iT'_{24}&=0\\ T'_{33}+iT'_{34}&=iT_{34}+T_{44}\\ T'_{43}+iT'_{44}&=iT_{44}-T_{34} \end{aligned} \tag{14a}$$

$$\begin{aligned} -T_{13}+iT_{14}&=0\\ -T_{23}+iT_{24}&=0\\ -T_{33}+iT_{34}&=iT'_{34}-T'_{44}\\ -T_{43}+iT_{44}&=iT'_{44}+T'_{34} \end{aligned} \tag{14b}$$

which are the relations between the components of the energy-momentum tensor for two observers moving with respect to each other with the velocity of light.

Consequently, in agreement with the asynchronous interpretation of relativistic dynamics and as for relations (13), we may interpret the left-hand side of (14a) and (14b) as the components of the energy-momentum density vector g^c_μ in frames moving with the velocity of light. The superscript c characterizes this situation. Moreover, if the primed frame is the wavefront and if an asterisk denotes quantities on the wavefront, we get from (14)

$$g^c_1 = g^c_2 = 0, \qquad g^c_4 = -ig^c_3 = T^*_{34} + iT^*_{34} \tag{15}$$

Of course g^c_μ may be written in a covariant form:

$$g^c_\mu = T_{\mu\nu} m_\nu \tag{16}$$

where m_ν is a null vector $(m_\nu m_\nu = 0)$ orthogonal to the wavefront. When the wavefront is the hyperplane $z - ct = 0$ one has $m_\nu = (0, 0, 1, -i)$.

Now if da is an invariant measure on the wavefront, the energy-momentum four-vector P^c_μ is given by the relation

$$P^c_\mu = \int g^c_\mu \, da \tag{17}$$

and to define the measure da, one has to take into account the following facts: (i) da must be invariant for any observer moving with the velocity of light, and (ii) the 3-volume measure on the light cone is zero (Synge, 1958). A beautiful illustration of this result may be found in Suffern (1988).

This suggests the following definition of da:

$$da = (\tfrac{1}{2}\, d\sigma_{\alpha\beta}\, d\sigma_{\alpha\beta})^{1/2}, \qquad d\sigma_{\alpha\beta} = \varepsilon_{\mu\nu\alpha\beta}\, dx^1_\mu\, dx^2_\nu \tag{18}$$

with

$$m_\nu\, dx^1_\nu = m_\nu\, dx^2_\nu = 0 \tag{18'}$$

where $\varepsilon_{\mu\nu\alpha\beta}$ is the permutation tensor, while dx^1_μ and dx^2_μ are two infinitesimal vectors orthogonal to m_μ. For instance, for the wavefront $z - ct = 0$ and $m_\mu = (0, 0, 1, -i)$ one has

$$dx^1_\mu = (dx, 0, 0, 0), \qquad dx^2_\mu = (0, dy, 0, 0)$$

and the measure da reduces to $dx\, dy$.

It is interesting to discuss the case of the electromagnetic field, whose energy-momentum tensor is given in the Appendix. Traditionally, one considers that the electromagnetic momentum density g_μ is given by the equality $g_j = s_j/c^2$, $g_4 = s_4$, that is, on the wavefront,

$$s^c_j = -icT^c_{j4}, \qquad s^c_4 = -T^c_{44} \tag{19}$$

while according to (15) one has

$$g^c_1 = g^c_2 = 0, \qquad g^c_3 = ig^c_4 = \frac{1}{c^2}(ics^c_4 - s^c_3) \tag{20}$$

The components g^c_1 and g^c_2 are zero as expected for any wave propagating along $0z$, which is not the case for s_1, s_2 except when the electromagnetic wave is a plane wave, as can be seen in the Appendix.

Consequently, except for plane waves, the Poynting vector does not represent the energy flow of the electromagnetic field, a result already emphasized by Rohrlich (1970), but which seems to have been overlooked. Moreover, making $v \mapsto c$ in Rohrlich's expression of the energy-momentum density vector leads to the relations (20).

4. CONCLUSIONS

The relations (13), which appear in a natural way as a consequence of the Lorentz transformation, promote the asynchronous interpretation of the energy-momentum density vector. Moreover, this interpretation makes it possible to take the limit leading to the relations (14).

This result is particularly interesting for discussing boundary value problems with data on a surface moving with the velocity of light (Hillion, 1988).

APPENDIX

The electromagnetic energy-momentum tensor $T_{\mu\nu}$ is (Synge, 1958)

$$T_{\mu\nu}=\frac{1}{4\pi}\left|\begin{matrix} E_x^2+H_x^2-W & E_xE_y+H_xH_y & E_xE_z+H_xH_z & i(E_yH_z-E_zH_y) \\ E_xE_y+H_xH_y & E_y^2+H_y^2-W & E_yE_z+H_yH_z & i(E_zH_x-E_xH_z) \\ E_xE_z+H_xH_z & E_yE_z+H_yH_z & E_z^2+H_z^2-W & i(E_xH_y-E_yH_x) \\ i(E_yH_z-E_zH_y) & i(E_zH_x-E_xH_z) & i(E_xH_y-E_yH_x) & -W \end{matrix}\right|$$

with $W=\frac{1}{2}(E^2+H^2)$. One sees at once that $s_1=s_2=0$ and $s_3=g_3$ only if $E_z=H_z=0$ and $|E|=|H|$.

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